Computational Complexity of Swish Is Solved

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Swish

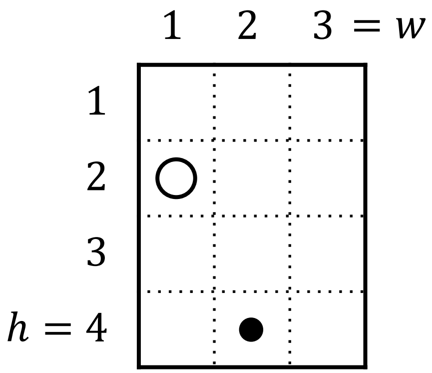
Swish

(Exactly one ○ and exactly one ● in the original game)

- Transparent and $(h \times w)$-grid shape including symbols ○ and ●

Input: $\mathcal{C}$: Set of **Cards**, $k \in \mathbf{Z}_{\geq 0}$

Goal: Test existence of **Swish** $\mathcal{S} \subseteq \mathcal{C}$ with $|\mathcal{S}| \geq k$



Card set with each cell in one of the following situations:

- Empty
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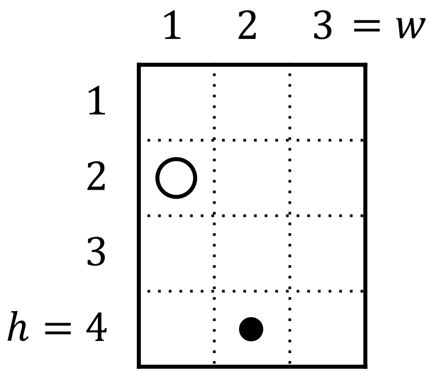

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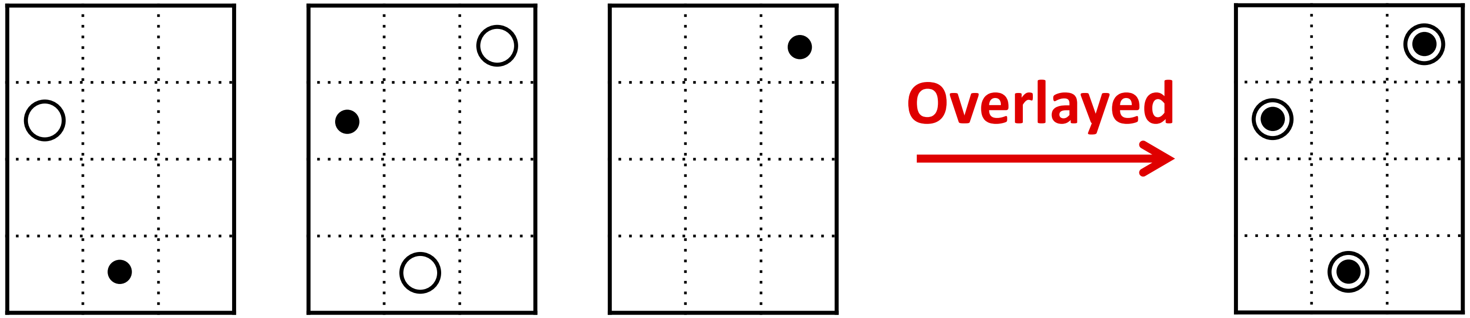

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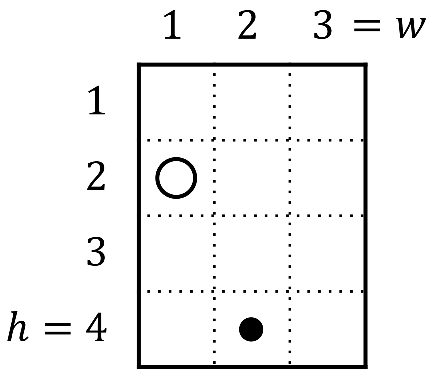
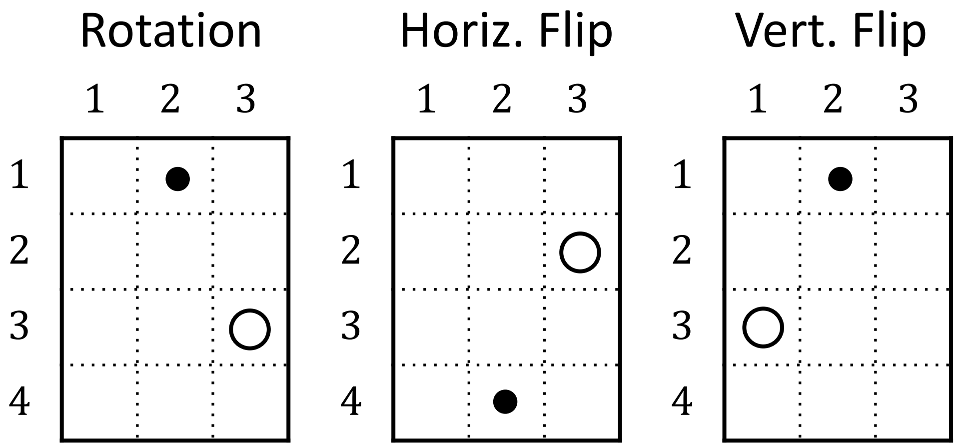


Each card is **transformable** as follows:

- Rotation: $(i, j) \mapsto (h - i + 1, w - j + 1)$
- Horizontal Flip: $(i, j) \mapsto (i, w - j + 1)$
- Vertical Flip: $(i, j) \mapsto (h - i + 1, j)$



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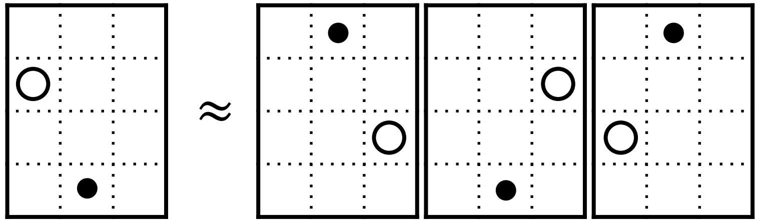

Thm.

- ∀Card has at most 1 symbol $\Longrightarrow$ Polytime
- ∃Card has at least 3 symbols $\Longrightarrow$ NP-complete

[Dailly, Lafourcade, Marcadet; FUN2024]

Thm. Suppose that ∀Card has (at most) 2 symbols

- Neither Rotation nor Flip is allowed $\Longrightarrow$ Polytime
- Rotation or Flip (or both) is allowed $\Longrightarrow$ NP-complete

[This work; FUN2026]

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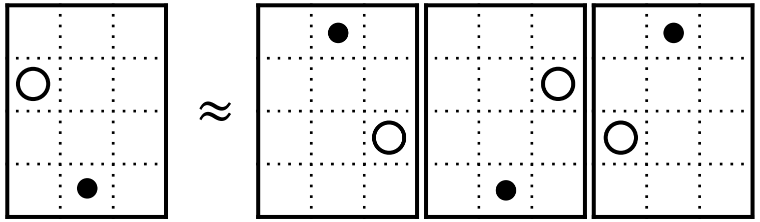

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Without Rotation or Flip (in P)

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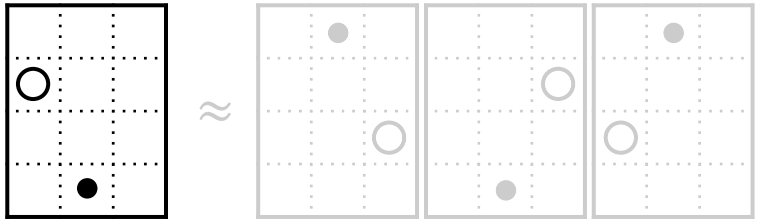

Reducible to Maximum-Weight Perfect Matching $\rightarrow$ $\mathrm{O}\left((hw)^3\right)$ time

- For each cell (i, j), prepare two vertices $v_{ij}^{\circ}, v_{ij}^{\bullet}$ connected by an edge of weight 0
- For each card, add an edge of weight 1 between the corresponding vertices

Lem. $\exists$Swish of size k $\Longleftrightarrow$ $\exists$Perfect Matching of weight k

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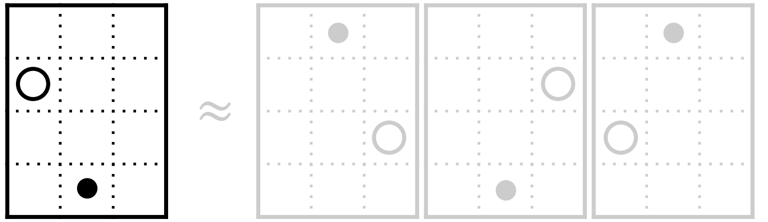

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Lem. $\exists$Swish of size k $\iff$ $\exists$Perfect Matching of weight k

Correspondence between Swish and PM

Proof of $\Longrightarrow$

- Swish of size k corresponds to a matching of weight k such that each pair of $v_{ij}^{\circ}, v_{ij}^{\cdot}$ is either both matched or both exposed
- Adding a 0-edge between each pair of exposed vertices makes it a PM

- For each cell (i, j), prepare two vertices $v_{ij}^{\circ}, v_{ij}^{\cdot}$ connected by an edge of weight 0
- For each card, add an edge of weight 1 between the corresponding vertices

Lem. $\exists$Swish of size k $\Longleftrightarrow$ $\exists$Perfect Matching of weight k

Correspondence between Swish and PM

Proof of $\Longleftarrow$

- Consider the matching obtained by removing all 0-edges from the PM
- Since each pair of $v_{ij}^{\circ}, v_{ij}^{\cdot}$ is still either both matched or both exposed and the weight is the same as the original PM, it corresponds to a Swish of size k

- For each cell (i, j), prepare two vertices $v_{ij}^{\circ}, v_{ij}^{\cdot}$ connected by an edge of weight 0
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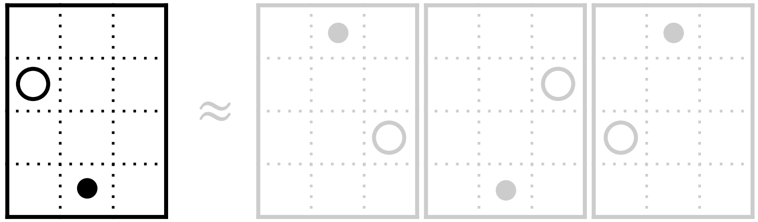

Reducible to Maximum-Weight Perfect Matching $\rightarrow$ $O\left((hw)^3\right)$ time

- For each cell (i, j), prepare two vertices $v_{ij}^{\circ}, v_{ij}^{\bullet}$ connected by an edge of weight 0
- For each card, add an edge of weight 1 between the corresponding vertices

[Remark]

- ∃Card has only one symbol $\rightarrow$ Duplicate the graph and use edge weights $0, 1, 2$
- Swish of size **Exactly** k $\rightarrow$ **Randomized** Polytime (cf. Exact Matching Problem)

With Rotation or Flip (NP-hard)

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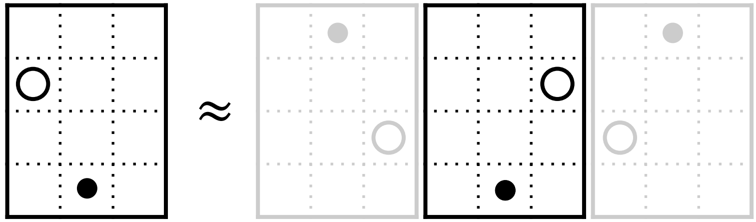
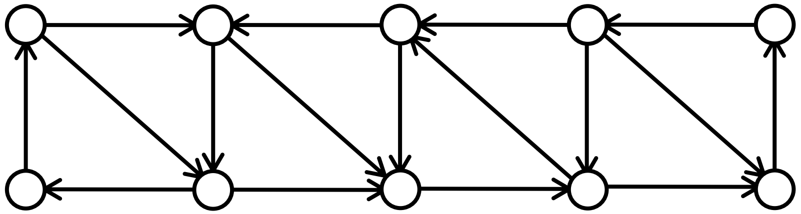


Lem.

Even Dicycle-Factor Problem (NP-hard) is polytime reducible to it

Input: $G = (V, E)$: Directed Graph

Goal: Test ∃**Even Dicycle-Factor**



[Bang-Jensen, Bessy 2014]
[Hörsch, Király, Mendoza-Cadena, Pap, Szabó, Y. 2025]

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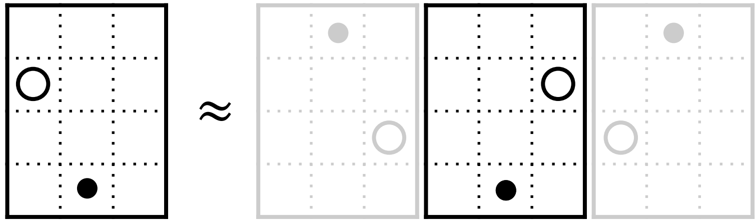

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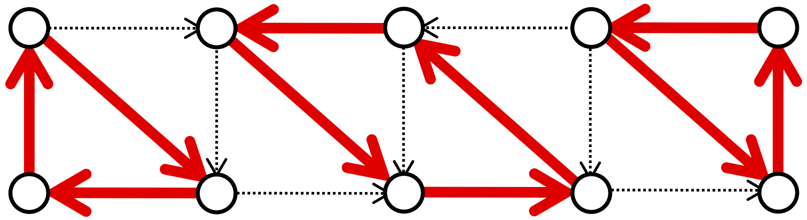

(This is in P by a reduction to a bipartite matching)

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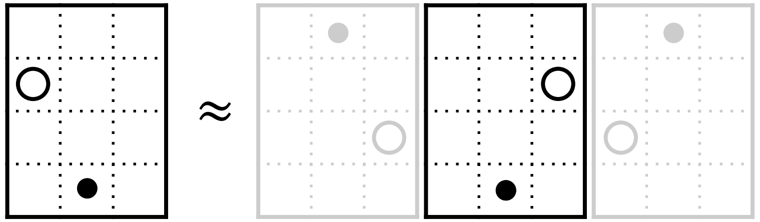

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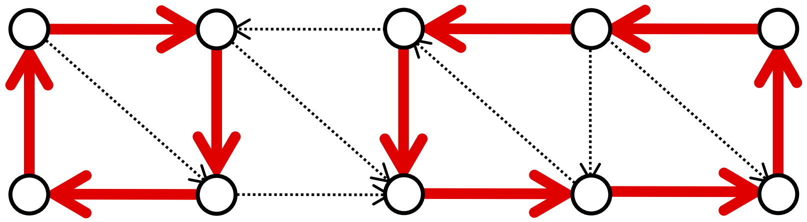

Even Dicycle-Factor $\rightarrow$ Swish with H-Flip

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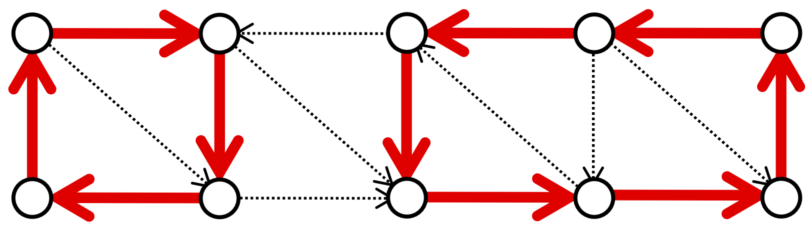
Goal: Test $\exists$**Even Dicycle-Factor**



Let $h = |V|$, $w = 4$, $k = hw$, and $V = \{v_1, v_2, \dots, v_h\}$

- For each vertex $v_i \in V$ and $j \in \{1, 2, 3\}$, prepare one card with ○ in cell (i, j) and ● in cell $(i, j + 1)$
- For each edge $(v_x, v_y) \in E$, prepare one card with ○ in cell $(x, 1)$ and ● in cell $(y, 1)$

Lem. $\exists$Even Dicycle-Factor $\Leftrightarrow$ $\exists$Swish of size $k = hw$

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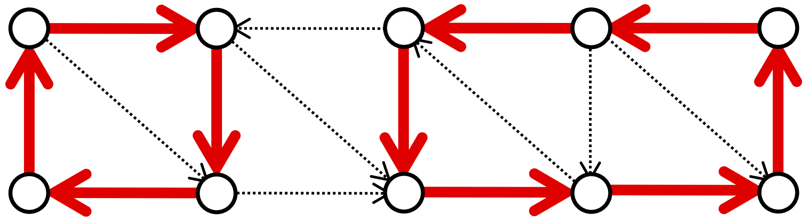

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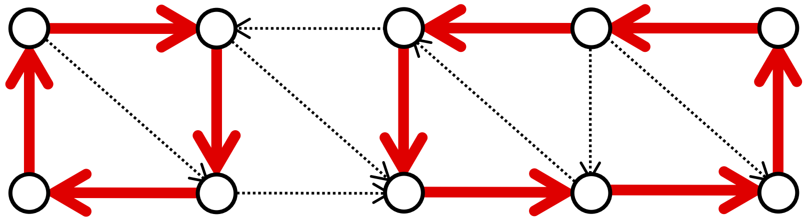

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<u>Lem.</u> ∃Even Dicycle-Factor $\iff$ ∃Swish of size $k = hw$

Correspondence between the solutions

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  - We can switch the symbols at $(i, 1)$ and $(i, 4)$ by Horizontal Flip
- For each edge $(v_x, v_y) \in E$,
  prepare one card with ○ in cell $(x, 1)$ and ● in cell $(y, 1)$
  - Transformable into $(x, 4)$ and $(y, 4)$ by Horizontal Flip
  - Fill the remaining symbols $\Longleftrightarrow$ Alternate 1, 4 and Close $\Longleftrightarrow$ Even Dicycles

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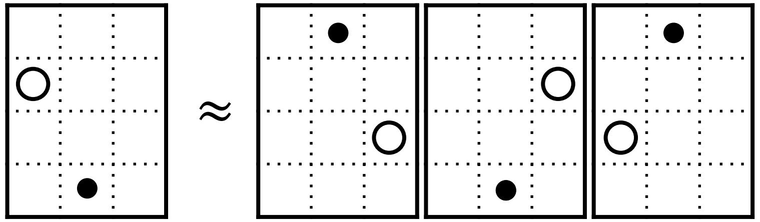

Lem.

Even Dicycle-Factor Problem (NP-hard) is polytime reducible to it

[Remark]

- The same reduction works for the other types by modifying the positions of symbols
- Even if multiple types of transformations are allowed, the same reduction works by arranging the symbols so that one of the transformations is meaningless

Summary

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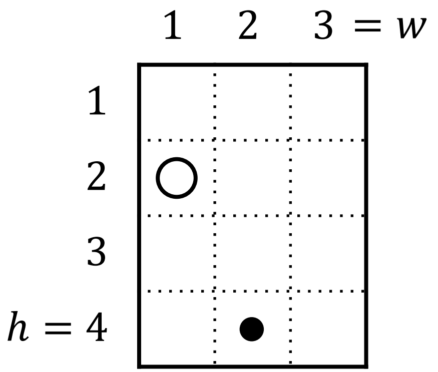

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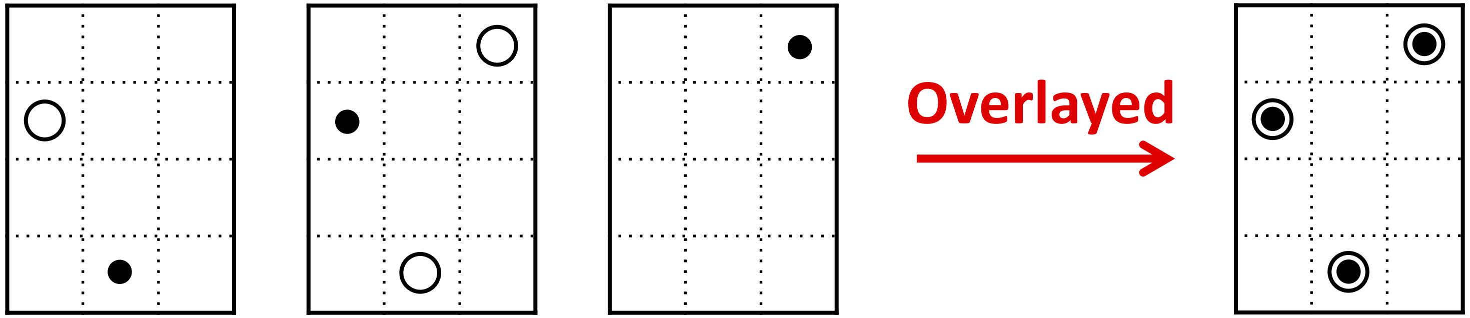

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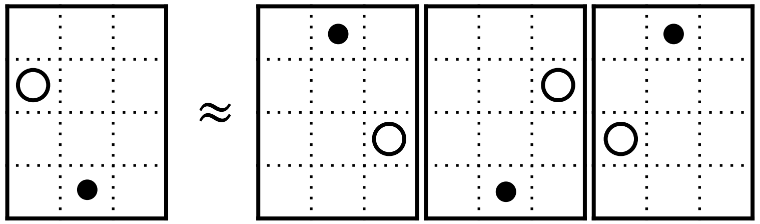

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